Sofic groups ....

Motivation: does there exist a non sofic group?

Easy fact: Every finite group embeds into S_n

Strategy: Amalgamated free products $G_1 *_H G_2$

Thm (Elek–Szabo inspired by Jung).

Any two sofic approximations of countable G are conjugate iff G is amenable.

Cor: (Elek–Szabo):

$G_1 *_H G_2$ is sofic if G_1, G_2 are sofic and H is amenable.

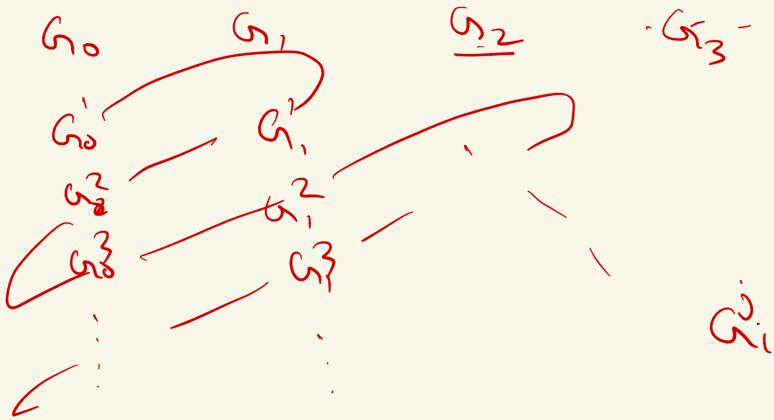
2-Steps: 1: construct random conjugate of the sofic embedding of G_2.

2: aligning the H.

Maybe: G iterative construction:

enumerate f.g subgroups G_i^j of G^j.

$G_0 = \mathbb{F}_2$.



$$G_1 = G *_{G_0} G$$

$$G_1' \subset G_1 \; ; \quad G_1 *_{G_{11}} G_1 = G_2$$

$$\rightsquigarrow \quad G_i \subset G_{i+1}$$

$$\bigcup G_i = G.$$

Goal: Understand how wild sofic embeddings of non amenable groups can be.

Thm (Hayes–KE): If G is initially subamenable and non amenable, then $\exists$ two sofic embeddings that are not conjugate by any automorphism.

Remark: proof uses von Neumann algebras crucially.

$\exists$ $G_1 *_H G_2$ which is not sofic.

Definitions:

Strategy of proof:

Thm: (Hayes–KE):
If G is initially subamenable and non amenable, then $\exists$ two sofic embeddings that are not conjugate by any automorphism.

Pf: step 1: construct an sofic embedding of G such that $G' \cap \mathcal{S} \curvearrowright \mathcal{L}$ Loeb measure space is ergodic.

$\prod_{n \to \mathcal{U}} D_n = L^\infty(X) \subset \prod_{n \to \mathcal{U}} M_n(\mathbb{C})$

$L^\infty(X) = \mathcal{L}$

$[\cdot]$ $[\cdot\ \cdot]$ $[\cdot\ \cdot\ \cdot]$ $[\cdot\ \cdot\ \cdot]$ ....
E_1 E_2 E_3

$(A_i)_{\mathcal{U}} = A \subset \prod_{n \to \mathcal{U}} E_i$

$A_i \subset E_i$

$\mu(A) = \lim_{n \to \mathcal{U}} \mu_i(A_i)$

Step 2: Construct an sofic embedding of G st

$G' \cap \mathcal{S} \curvearrowright \mathcal{L}$ is not ergodic.

Convex structure of sofic embeddings. (Paunescu)

$$P \prod_{n \to \mathcal{U}} S_n \quad P \qquad \lambda \pi_1 \oplus \lambda_2 \pi_2$$

$$\begin{array}{ccccc} E_1 & E_2 & E_3 & & \\ (\cdot) & (\cdot\ \cdot) & (\cdot\ \cdot\ \cdot) & (\cdot\ \cdot\ \cdot\ \cdot) & (\cdot \cdots) \\ \| & \| & \| & & \\ E_1^1 & E_1^2 & E_1^3 & & \\ \sqcup & \sqcup & \sqcup & & \\ E_2^1 & E_2^2 & E_2^3 & & \end{array}$$

$$\boxed{\lim_{j \to \mathcal{U}} \mu(E_j^i) > 0}$$

Step 3:

$\pi_1(G)' \curvearrowright \mathcal{S}$ is ergodic

$\iff \pi_2(G)' \curvearrowright \mathcal{S}$ is ergodic

where π_1 and π_2 are <u>automorphically conjugate</u>

<u>sketch:</u>

$$\mathcal{S} \subset \prod_{n \to \mathcal{U}} M_n(\mathbb{C})$$

$$\boxed{\begin{array}{l} L^\infty(X) \subset W^*(\mathcal{S}) \\ \ \| \\ L^\infty(\mathcal{L}) \end{array}} \subset \prod_{n \to \mathcal{U}} M_n(\mathbb{C})$$

$T_n \xrightarrow{WOT} T$ if $\langle T_n \xi, \eta \rangle \to \langle T\xi, \eta \rangle$ $\forall \xi, \eta \in \mathcal{H}$.

$\mathbb{1}_{(E_i)_n} \in W^*(\mathcal{S})$.

$g^n \xrightarrow{WOT} \mathbb{1}_{(E_i)_n}$

Sym_n: finite symmetric group of rank n.
Recall normalized Hamming distance which is a bi-invariant metric on Sym_n:

$$d_n(\sigma, \rho) = \frac{|\{i \mid \sigma(i) \neq \rho(i)\}|}{n}.$$

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A sequence of maps $\sigma_n \colon G \to \mathrm{Sym}_n$ is said to be an approximate homomorphism if for all $g, h \in G$ we have

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Definition

For the results in Section 3 we will need the notion of ultraproducts of groups with bi-invariant metrics. Let ω be a free ultrafilter on $\mathbb{N}$. Let (G_n, d_n) be countable groups with bi-invariant metrics. Denote by

$$\prod_{n\to\omega} (G_n, d_n) = \frac{\{(g_n)_{n\in\mathbb{N}}\}}{\{(g_n) | \lim_{n\to\omega} d_n(g_n, 1_{G_n}) = 0\}}.$$

Let $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$ be a non principal ultrafilter on $\mathbb{N}$. Denote by $\mathcal{S} := \prod_{n \to \omega} (\mathrm{Sym}_n, d_n)$. We call $\mathcal{S}$ a universal sofic group.

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Denote by χ the trace on $\mathcal{S}$, given by $\chi((p_n)_{n\to\omega}) = 1 - \lim_{n\to\omega} d_n(1, p_n)$.

Definition

Say that G is sofic if it admits a sofic approximation, equivalently, there exists an injective homomorphism $\pi : G \to \mathcal{S}$ such that $\chi(\pi(g)) = \delta_{g=e}$.

A group is initially subamenable if for all $F \subseteq G$ finite, there is an amenable group H and an injective map $\phi\colon F \to H$ so that $\phi(xy) = \phi(x)\phi(y)$ for all pairs $(x, y) \in F \times F$ with the property that $xy \in F$ (i.e. ϕ is a homomorphism "when it makes sense to be").

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Equivalently, a group G is initially subamenable if it is the limit of a sequence of amenable groups in the space of marked groups. This is a large family of sofic groups containing all residually finite groups. Gromov asked if all sofic groups are initially subamenable and this was answered in the negative by Cornulier.

Two sofic embedddings π_1, π_2 of G into a universal sofic group $\mathcal{S}$ are automorphically conjugate if there is an automorphism $\Phi \in \mathrm{Aut}(\mathcal{S})$ so that $\Phi \circ \pi_1 = \pi_2$. We say that G satisfies the generalized Elek-Szabo property if any two sofic embeddings π_1, π_2 of G into a universal sofic group $\mathcal{S}$ are automorphically conjugate.

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Theorem (Hayes-KE '23)

Let G be an initially subamenable group. Then G is amenable iff any two sofic embeddings π_1, π_2 of G into a universal sofic group $\mathcal{S}$ are automorphically conjugate.